Sensors and Materials, Vol. 34, No. 3 (2022) 1211–1219
MYU Tokyo

S & M 2884

Chaos Synchronization via Ameliorated Dynamic Control

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(Received June 29, 2021; accepted January 12, 2022)

Keywords: controlled synchronization, dynamic control, Lyapunov theory

Although the synchronization scheme of dynamic control proposed by previous studies can synchronize chaotic systems, not all chaotic systems can be successfully synchronized. In this paper, we propose an ameliorated synchronization scheme of dynamic control, in which the controllers are changed from 1D to 2D and are added to two signals of the slave system. As a result, the synchronization of chaotic systems that fail to be synchronized is successfully achieved. The synchronous stability is investigated by using Lyapunov theory and the master stability function approach. Two applications of Lü and Lü-like chaotic systems are compared. The results show that the proposed ameliorated scheme is effective and can be used to develop a chaos-synchronization sensor.

1. Introduction

For many studies on chaotic systems, synchronization is an applied and attractive research topic. In some cases, to achieve synchronization, specific constraints need to be imposed, and this phenomenon is called forced or controlled synchronization. Recently, there has been increasing interest in this type of research, various synchronization modes and methods have been studied,(1–7) and a large number of technical applications have been produced.(8–11)

Researchers have been pursuing global synchronization methods suitable for all systems. However, the applicability of some synchronization schemes is limited. For example, in a previous study, a synchronization scheme with a static controller failed to synchronize the Rössler system.(12) To address this limitation, Ramirez *et al.*(12) proposed a scheme with a dynamic controller composed of a first-order system instead of the classical static controller, and demonstrated that their proposed synchronization strategy was applicable to a large class of dynamical systems including chaotic systems. However, is it really applicable to all chaotic systems?

We reexamined the scheme and found that it was suitable for most dynamical systems but not all chaotic systems. For example, it failed to synchronize Lü-like chaotic systems.(13,14) To

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https://doi.org/10.18494/SAM3490

ISSN 0914-4935

overcome this problem, in this study, we proposed an ameliorated synchronization scheme in which two dynamic controllers driven by the scaled difference between the signals of master and slave systems were designed. In addition, to enhance the coupling between master and slave systems, two state variables were measured in the master and slave systems, and the two controllers were imported into two different signals of the slave system. A pair of Lü chaotic systems and a pair of Lü-like chaotic systems were considered as examples, which verified the validity of the proposed synchronization scheme. The master stability function(15) and the Lyapunov indirect method(16) were used to investigate the local stability of the error function. Moreover, the results of this study are applicable to the development of a chaos-synchronization sensor.

This paper is organized as follows. First, Sect. 2 presents the synchronization scheme with a dynamic controller. The proposed ameliorated synchronization scheme with dynamic controllers is introduced in Sect. 3. In Sect. 4, two examples, Lü chaotic systems and Lü-like chaotic systems, are presented. Finally, a discussion of the obtained results and some conclusions are provided in Sect. 5.

2. Synchronization Scheme with a Dynamic Controller

Consider the following master–slave systems.

$$\boldsymbol{Master}:\begin{cases}\dot{x}_m = F\left(x_m\right)\\ y_m = x_m\end{cases} \tag{1}$$

$$\boldsymbol{Slave}:\begin{cases}\dot{x}_s = F(x_s) - Bh\\ y_s = Cx_s\end{cases} \tag{2}$$

$$\boldsymbol{Dynamic\,controller}:\left\{\dot{h} = -\alpha h - kC\left(x_m - x_s\right)\right. \tag{3}$$

Here, $x_m, x_s \in \mathbb{R}^n$ are the state vectors of the master and slave systems, respectively, $y_i \in \mathbb{R}$, $i = m, s$ are the corresponding outputs, function F is assumed to be sufficiently smooth, $B \in \mathbb{R}^n$ is a constant column vector, $C \in \mathbb{R}^{1\times n}$ is a constant row vector, $h \in \mathbb{R}$ is the dynamic control input, $k \in \mathbb{R}_+$ is the coupling strength, and $\alpha \in \mathbb{R}_+$ is a design parameter.

Assume that the nonlinear function F consists of linear and nonlinear parts:

$$F(x_i) = Ax_i + f(x_i), i = m, s, \tag{4}$$

where $A \in \mathbb{R}^{n\times n}$ is a constant matrix.

Then, the error dynamics for the systems in Eqs. (1)–(3) is obtained as

$$\begin{bmatrix} \dot{e} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} A & B \\ -kC & -\alpha \end{bmatrix} \begin{bmatrix} e \\ h \end{bmatrix} + \begin{bmatrix} g(t,e) \\ 0 \end{bmatrix}, \tilde{e} = \begin{bmatrix} \dot{e} \\ \dot{h} \end{bmatrix}, \overline{A} = \begin{bmatrix} A & B \\ -kC & -\alpha \end{bmatrix} \begin{bmatrix} e \\ h \end{bmatrix}, \overline{g}(t,\tilde{e}) = \begin{bmatrix} g(t,e) \\ 0 \end{bmatrix}, \tag{5}$$

where $e = x_m - x_s$, $g(t, e) = f(x_m) - f(x_s)$, $\tilde{e} \in \mathbb{R}^{n+1}$ is the state vector, and matrix $\overline{A} \in \mathbb{R}^{(n+1)\times(n+1)}$ is assumed to be Hurwitz.[17] Since the trajectories of the master system are bounded, the term $\overline{g}(t,\tilde{e})$ can be regarded as a perturbation that will vanish on e if it satisfies

$$\| \overline{g}(t,\tilde{e}) \|_2 \leq \gamma \| \tilde{e} \|_2, \ \forall t \geq 0, \forall \tilde{e} \in D \subset \mathbb{R}^{n+1}, \tag{6}$$

where $\| \cdot \|_2$ denotes the Euclidean norm. The stability properties of the error dynamics in Eq. (5) can be inspected as follows. First, consider the quadratic Lyapunov function

$$V(\tilde{e}) = \tilde{e}^T P \tilde{e}, \tag{7}$$

where $P \in \mathbb{R}^{(n+1)\times(n+1)}$ is a positive definite and symmetric matrix that is the solution of the Lyapunov equation

$$P\overline{A} + \overline{A}^T P = -Q, \tag{8}$$

where $Q \in \mathbb{R}^{(n+1)\times(n+1)}$ is a positive definite and symmetric matrix: a standard choice is $Q = I$, where I is the identity matrix of appropriate dimensions. In addition, a unique solution for Eq. (8), $P = P^T > 0$, always exists because $\overline{A}$ in Eq. (5) has been assumed to be Hurwitz.

Next, through calculations, the time derivative of the Lyapunov function in Eq. (7) satisfies

$$\dot{V}(\tilde{e}) \leq -[\lambda_{min}(Q) - 2\lambda_{max}(P)\gamma] \| \tilde{e} \|_2^2, \tag{9}$$

where $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ denote the minimum and maximum eigenvalues, respectively.

In case that $\overline{A}$ is assumed to be Hurwitz, a sufficient condition for the local stability of the system in Eq. (5) is that the bound γ on the perturbation term in Eq. (6) is sufficiently small to satisfy

$$\gamma < \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)}. \tag{10}$$

Consequently, the time derivative of the Lyapunov function is not greater than zero, that is, the error dynamics is asymptotically stable and the master and slave systems synchronize.

Most pairs of systems can indeed synchronize through the above scheme, but not all systems. For example, the synchronization scheme with a dynamic controller fails to synchronize Lü-like chaotic systems. To overcome this problem, we propose an ameliorated synchronization scheme with dynamic controllers.

3. Ameliorated Synchronization Scheme with Dynamic Controllers

Consider the following master–slave systems.

$$\boldsymbol{Master}:\begin{cases}\dot{x}_m = F(x_m)\\ y_m = x_m\end{cases} \tag{11}$$

$$\boldsymbol{Slave}:\begin{cases}\dot{x}_s = F(x_s) - D\begin{bmatrix}h_1\\ h_2\end{bmatrix}\\ y_s = Cx_s\end{cases} \tag{12}$$

$$\boldsymbol{Dynamic\,controllers}:\left\{\begin{bmatrix}\dot{h}_1\\ \dot{h}_2\end{bmatrix} = -\alpha\begin{bmatrix}h_1\\ h_2\end{bmatrix} - kE(x_m - x_s)\right. \tag{13}$$

Here, $x_m, x_s \in \mathbb{R}^n$ are the state vectors of the master and slave systems, respectively, $y_i \in \mathbb{R}$, $i = m$, s are the corresponding outputs, function F is assumed to be sufficiently smooth, $D \in \mathbb{R}^{n\times 2}$ and $E \in \mathbb{R}^{2\times n}$ are constant vectors, h_1 and h_2 are the dynamic controllers, α is a design parameter, and k is the coupling strength.

Since the underlying theory is the same and the calculation is similar except that the error dynamics dimension is increased and the calculation is more complicated, the following process is omitted. The predominant feature of our proposed ameliorated synchronization scheme is that two signals of the slave system are input to the controllers at the same time, which simultaneously increases the coupling between the signals and successfully synchronizes Lü-like chaotic systems that are 'nonsynchronizable' when using the previous synchronization scheme.

4. Application Examples

4.1 Synchronization scheme with a dynamic controller

4.1.1 Example 1: Lü chaotic systems

Lü chaotic systems,[18] which are important models of 3D chaotic systems, are taken as an example and described as follows.

$$\textbf{\textit{Master}}:\begin{cases}\dot{x}_m = a(y_m - x_m)\\ \dot{y}_m = -x_m z_m + c y_m\\ \dot{z}_m = x_m y_m - b z_m\end{cases} \tag{14}$$

$$\textbf{\textit{Slave}}:\begin{cases}\dot{x}_s = a(y_s - x_s)\\ \dot{y}_s = -x_s z_s + c y_s - h\\ \dot{z}_s = x_s y_s - b z_s\end{cases} \tag{15}$$

$$\textbf{\textit{Dynamic controllers}}:\dot{h} = -\alpha h - k(y_m - y_s)\,, \tag{16}$$

where $(a, b, c) = (36, 3, 20)$. When the error function is set to $e_x = x_m - x_s$, $e_y = y_m - y_s$, and $e_z = z_m - z_s$, the dynamics can be written in the form of Eq. (5) with

$$\overline{A} = \begin{bmatrix} -36 & 36 & 0 & 0\\ 0 & 20 & 0 & 1\\ 0 & 0 & -3 & 0\\ 0 & -k & 0 & -\alpha \end{bmatrix} \text{ and } \overline{g}(t,\tilde{e}) = \begin{bmatrix} 0\\ -x_m z_m + x_s z_s\\ x_m y_m - x_s y_s\\ 0 \end{bmatrix}. \tag{17}$$

The characteristic polynomial of matrix $\overline{A}$ in Eq. (17) is given by

$$p(\lambda) = (\lambda + 36)(\lambda + 3)[\lambda^2 + (\alpha - 20)\lambda + (k - 20\alpha)]. \tag{18}$$

According to the Rouih–Hurwitz stability criterion,[15] Eq. (18) will have negative roots if and only if the following condition is satisfied:

$$\alpha > 20 \text{ and } k > 20\alpha. \tag{19}$$

Consequently, the error dynamics of the systems in Eqs. (14)–(16) is globally asymptotically stable, that is, the master and slave Lü chaotic systems will asymptotically synchronize, as shown in Fig. 1.

4.1.2 Example 2: Lü-like chaotic systems

Next, consider Lü-like chaotic systems as another example, described as follows.

$$\textbf{\textit{Master}}:\begin{cases}\dot{x}_m = a(y_m - x_m) + d x_m z_m\\ \dot{y}_m = f y_m - x_m z_m\\ \dot{z}_m = c z_m + x_m y_m - e x_m^2\end{cases} \tag{20}$$

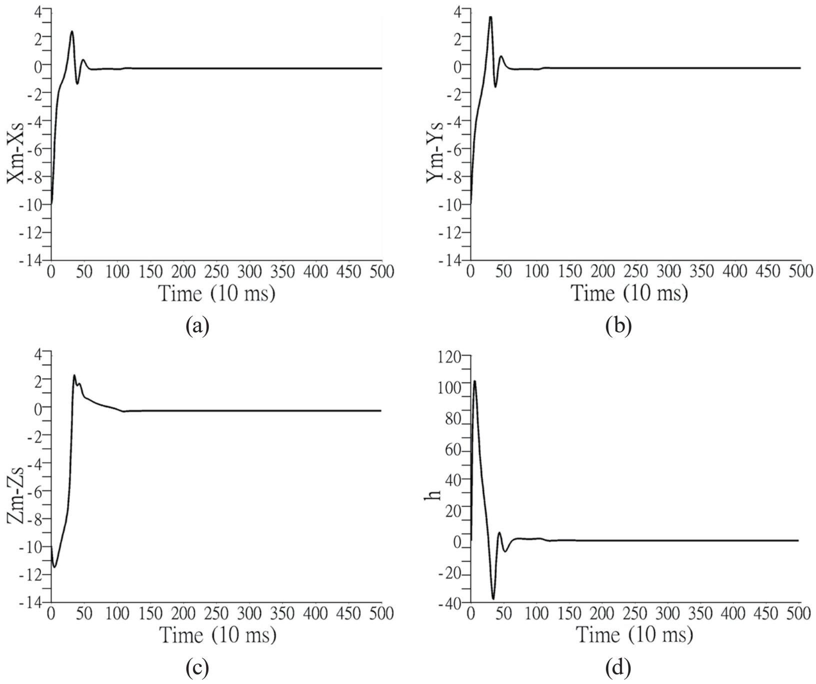

Fig. 1. Time series of (a) $x_m - x_s$, (b) $y_m - y_s$, (c) $z_m - z_s$, and (d) h of error dynamics of Lü chaotic systems with parameters $x_m(0) = y_m(0) = z_m(0) = 0.1$, $x_s(0) = y_s(0) = z_s(0) = 10$, $h(0) = 0$, $\alpha = 21$, and coupling strength $k = 440$ underlying the synchronization scheme with a dynamic controller.

$$\textbf{\textit{Slave}}:\begin{cases}\dot{x}_s = a(y_s - x_s) + dx_s z_s \\ \dot{y}_s = fy_s - x_s z_s - h \\ \dot{z}_s = cz_s + x_s y_s - ex_s^2\end{cases} \tag{21}$$

$$\textbf{\textit{Dynamic controllers}}: \dot{h} = -\alpha h - k(y_m - y_s)\,, \tag{22}$$

where $(a, c, d, e, f) = (40, 5/6, 0.5, 0.65, 20)$. When the error function is set to $e_x = x_m - x_s$, $e_y = y_m - y_s$, and $e_z = z_m - z_s$, the dynamics can be written in the form of Eq. (5) with

$$\bar{A} = \begin{bmatrix} -40 & 40 & 0 & 0 \\ 0 & 20 & 0 & 1 \\ 0 & 0 & 5/6 & 0 \\ 0 & -k & 0 & -\alpha \end{bmatrix} \text{ and } \bar{g}(t,\tilde{e}) = \begin{bmatrix} 0.5(x_m z_m - x_s z_s) \\ -x_m z_m + x_s z_s \\ x_m y_m - x_s y_s - 0.65(x_m^2 - x_s^2) \\ 0 \end{bmatrix}. \tag{23}$$

The characteristic polynomial of matrix $\bar{A}$ in Eq. (23) is given by

$$p(\lambda) = (1/6)(\lambda + 40)[\lambda^3 + (6\alpha - 125)\lambda^2 + (6k - 125\alpha + 100)\lambda + (100\alpha - 5k)]\,. \tag{24}$$

It is clear from Eq. (24) that no values of k and $\alpha \in \mathbb{R}$ exist such that matrix $\bar{A}$ in Eq. (23) can be converted to a Hurwitz matrix. That is, the synchronization scheme with a dynamic controller fails to synchronize Lü-like chaotic systems.

4.2 Proposed ameliorated synchronization scheme with dynamic controllers

To find a global synchronization method to solve the above situation that synchronization fails to be achieved, we propose an ameliorated synchronization scheme with dynamic controllers. We reconsider Lü-like chaotic systems as an example described as

$$\textbf{\textit{Master}}:\begin{cases}\dot{x}_m = a(y_m - x_m) + dx_m z_m,\\ \dot{y}_m = fy_m - x_m z_m,\\ \dot{z}_m = cz_m + x_m y_m - ex_m^2,\end{cases} \tag{25}$$

$$\textbf{\textit{Slave}}:\begin{cases}\dot{x}_s = a(y_s - x_s) + dx_s z_s,\\ \dot{y}_s = fy_s - x_s z_s - h_1,\\ \dot{z}_s = cz_s + x_s y_s - ex_s^2 - h_2,\end{cases} \tag{26}$$

$$\textbf{\textit{Dynamic controllers}}:\begin{cases}\dot{h}_1 = -\alpha h_1 - k(y_m - y_s),\\ \dot{h}_2 = -\alpha h_2 - k(z_m - z_s),\end{cases} \tag{27}$$

where $(a, c, d, e, f) = (40, 5/6, 0.5, 0.65, 20)$. When the error function is set to $e_x = x_m - x_s$, $e_y = y_m - y_s$, and $e_z = z_m - z_s$, the dynamics can be written in the form of Eq. (5) with

$$\bar{A} = \begin{bmatrix} -40 & 40 & 0 & 0 & 0\\ 0 & 20 & 0 & 1 & 0\\ 0 & 0 & 5/6 & 0 & 1\\ 0 & -k & 0 & -\alpha & 0\\ 0 & 0 & -k & 0 & -\alpha \end{bmatrix} \text{ and } \bar{g}(t,\tilde{e}) = \begin{bmatrix} 0.5(x_m z_m - x_s z_s)\\ -x_m z_m + x_s z_s\\ x_m y_m - x_s y_s - 0.65(x_m^2 - x_s^2)\\ 0\\ 0 \end{bmatrix}. \tag{28}$$

The characteristic polynomial of matrix $\bar{A}$ in Eq. (28) is given by

$$\begin{aligned} p(\lambda) = (1/6)(\lambda+40)[6\lambda^4 + (12\alpha - 125)\lambda^3 + \left(6\alpha^2 - 250\alpha + 12k + 100\right)\lambda^2 \\ + \left(200\alpha - 125\alpha^2 + 12\alpha k - 125k\right)\lambda + \left(100\alpha^2 + 6k^2 - 125\alpha k\right)] \end{aligned} \tag{29}$$

According to the Rouih–Hurwitz stability criterion, the error dynamics of the systems in Eqs. (25)–(27) is globally asymptotically stable if the following condition is satisfied:

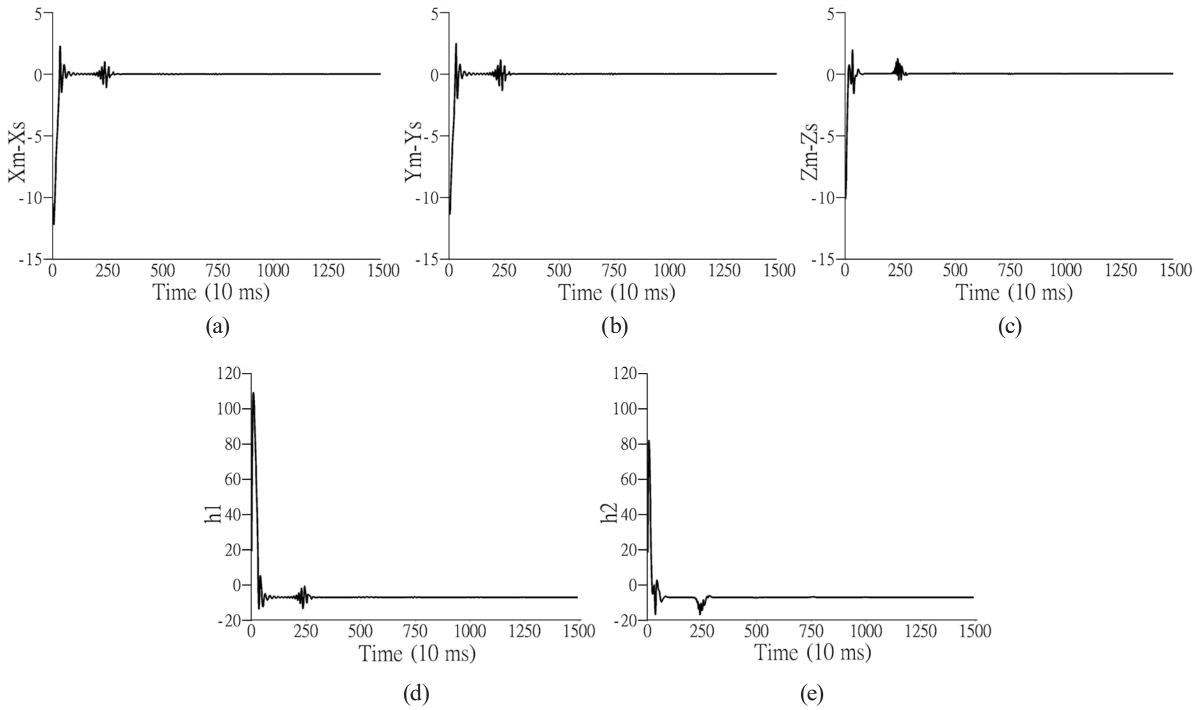

Fig. 2. Time series of (a) $x_m - x_s$, (b) $y_m - y_s$, (c) $z_m - z_s$, (d) h_1, and (e) h_2, of error dynamics of Lü-like chaotic systems with parameters $x_m(0) = y_m(0) = z_m(0) = 0.1$, $x_s(0) = y_s(0) = z_s(0) = 10$, $h_1(0) = h_2(0) = 0$, $\alpha = 21$, , and coupling strength $k = 440$ underlying the proposed ameliorated synchronization scheme with dynamic controllers.

$$\alpha > 20.9 \text{ and } k > 20\alpha. \tag{30}$$

Consequently, the proposed ameliorated synchronization scheme with dynamic controllers successfully synchronizes Lü-like chaotic systems, as shown in Fig. 2.

5. Discussion

Although the synchronization scheme with a dynamic controller can indeed induce synchronization in some cases where the standard master–slave scheme with the static controller fails to synchronize the systems, it cannot synchronize all systems. Therefore, we propose an ameliorated synchronization with dynamic controllers to solve this predicament. The potential of our proposed ameliorated synchronization scheme is that the coupling between systems is enhanced in dimensionality. Owing to the characteristics of certain systems, even if the coupling strength of systems with the same dimension is increased, the systems cannot be synchronized. In such a case, according to the Rouih–Hurwitz stability criterion, it is impossible to find values of design parameter α and coupling strength k that make the error dynamics of systems asymptotically stable. To further enhance the overall coupling effect on systems, our proposed ameliorated scheme enhances the coupling strength in dimensionality, which means importing the coupling in two different dimensions at the same time. The result has the effect of synchronization as expected.

6. Conclusions

To seek a global method to synchronize all systems, an ameliorated synchronization scheme with dynamic controllers is proposed, which successfully synchronizes Lü-like chaotic systems that cannot be synchronized originally by enhancing the coupling in dimensionality. Our results verify the feasibility and effectiveness of our proposed synchronization scheme, which can be used to develop a chaos-synchronization sensor for complex systems.

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