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Ameliorated Dynamic Control Scheme for Synchronization of Distinct Chaotic Systems and Stability Analysis

Chih-Hung Tsai,[1] Yan-Lin Zhong,[1] Ming-Chi Lu,[1]
Chen-An Chan,[1] Chia-Ju Liu,[2] and Ming-Chung Ho[1*]

[1]Department of Physics, National Kaohsiung Normal University,
No. 62, Shenjhong Rd., Yanchao District, Kaohsiung City 824, Taiwan, ROC
[2]Graduate Institute of Science Education & Environmental Education, National Kaohsiung Normal University,
No. 62, Shenjhong Rd., Yanchao District, Kaohsiung City 824, Taiwan, ROC

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The dynamic control synchronization schemes proposed in previous studies can synchronize chaotic systems with different initial values of the same type; however, not all chaotic systems can achieve synchronization. Therefore, previous synchronization schemes have been improved. In this study, we investigated the synchronization of two entirely different chaotic systems using the ameliorated dynamic control scheme, where a new controller design is integrated into the system signals to achieve synchronization between distinct systems. The stability of synchronization between the two different chaotic systems is analyzed using the Lyapunov theory and the master stability function approach. The results confirm that the previously proposed improved scheme is effective and can be utilized in the development of chaotic synchronization decrypters.

1. Introduction

Synchronization is a widely studied and fascinating topic in the field of chaotic systems. In some instances, specific constraints must be applied to achieve synchronization, a process referred to as forced or controlled synchronization. In recent years, there has been an increasing interest in this area of research, leading to the exploration of various synchronization techniques and modes,(1–7) which have resulted in numerous technical applications.(8–11)

A global synchronization method applicable for all systems has long been the goal of researchers. However, some synchronization schemes have restricted applicability. For instance, in a previous study, a synchronization scheme that utilized a static controller was unsuccessful in synchronizing the Rössler system.(12) To overcome this limitation, Ramirez *et al.*(12) introduced a scheme featuring a dynamic controller made up of a first-order system, as opposed to the traditional static controller, and showed that their synchronization strategy was effective for a wide range of dynamical systems, including chaotic ones. However, does this strategy really apply to all chaotic systems?

*Corresponding author: e-mail: t1603@mail.nknu.edu.tw
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After reevaluating the scheme, we discovered that while it was appropriate for most dynamical systems, it was not applicable to all chaotic systems. For instance, it cannot synchronize the Lorenz and Lü chaotic systems.[13] In this study, we propose an improved synchronization scheme to address this issue. We designed a dynamic controller driven by the proportional differences between the master and slave system signals. By simultaneously measuring the state variables in both the master and slave systems and introducing the dynamic controller into the slave system's signals, we enhance the coupling between the two systems. The Lorenz and Lü chaotic systems were used as a test case in this study, employing the master stability function[14] along with the Lyapunov indirect method[15] to examine the partial stability of the error function, confirming that the proposed synchronization scheme is effective. The results of this research can be applied to the development of chaotic synchronization sensors.

2. Synchronization Scheme Using a Dynamic Controller

Consider the master–slave system described below.

$$\textbf{Master}:\begin{cases}\dot{x}_m = F\left(x_m\right)\\ y_m = x_m\end{cases} \tag{1}$$

$$\textbf{Slave}:\begin{cases}\dot{x}_s = F\left(x_s\right) - Bh\\ y_s = Cx_s\end{cases} \tag{2}$$

$$\textbf{Dynamic controllers}:\left\{\dot{h} = -\alpha h - kC\left[x_m - x_s\right]\right. \tag{3}$$

In this context, $x_m \in \mathbb{R}^n$ represents the state vector of the master system, whereas $x_s \in \mathbb{R}^n$ denotes the state vector of the slave system. The outputs of the two systems are $y_m \in \mathbb{R}$ and $y_s \in \mathbb{R}$, respectively. It is assumed that function F is sufficiently smooth, B is a constant column vector in $\mathbb{R}^n$, C is a constant row vector in $\mathbb{R}^{1\times n}$, h is the dynamic control input in $\mathbb{R}$, k is the coupling strength in $\mathbb{R}_+$, and α is a design parameter in $\mathbb{R}_+$.

Suppose that the nonlinear function F comprises both linear and nonlinear components:

$$F\left(x_i\right) = Ax_i + f\left(x_i\right), i = m, s, \tag{4}$$

where A is a constant matrix in $\mathbb{R}^{n\times n}$.

Thus, the error dynamics for the systems described in Eqs. (1)–(3) can be expressed as

$$\begin{pmatrix}\dot{e}\\ \dot{h}\end{pmatrix} = \begin{pmatrix}A & B\\ -kC & -\alpha\end{pmatrix}\begin{pmatrix}e\\ h\end{pmatrix} + \begin{pmatrix}g\left(t,e\right)\\ 0\end{pmatrix},\ \tilde{e} = \begin{pmatrix}\dot{e}\\ \dot{h}\end{pmatrix},\ \overline{A} = \begin{pmatrix}A & B\\ -kC & -\alpha\end{pmatrix}\begin{pmatrix}e\\ h\end{pmatrix},\ \overline{g}\left(t,\tilde{e}\right) = \begin{pmatrix}g\left(t,e\right)\\ 0\end{pmatrix}, \tag{5}$$

where $e = x_m - x_s$, $g(t,e) = f(x_m) - f(x_s)$, $\tilde{e} \in \mathbb{R}^{n+1}$ is the state vector, and matrix $\overline{A} \in \mathbb{R}^{(n+1)\times(n+1)}$ is assumed to be Hurwitz.[16] Because the paths of the master system are confined within certain limits, the term $\overline{g}(t,\tilde{e})$ can be considered a perturbation that will disappear on e if it meets the following condition:

$$\left\|\overline{g}(t,\tilde{e})\right\|_2 \le \gamma \left\|\tilde{e}\right\|_2, \ \forall t \ge 0, \ \forall \tilde{e} \in D \subset \mathbb{R}^{n+1}, \tag{6}$$

where $\|\bullet\|_2$ represents the Euclidean norm. The stability characteristics of the error dynamics in Eq. (5) can be analyzed as follows. First, let us examine the quadratic Lyapunov function.

$$V(\tilde{e}) = \tilde{e}^T P \tilde{e} \tag{7}$$

Here, $P \in \mathbb{R}^{(n+1)\times(n+1)}$ is a symmetric matrix that is positive definite and acts as the solution to the Lyapunov equation.

$$P\overline{A} + \overline{A}^T P = -Q \tag{8}$$

Here, $Q \in \mathbb{R}^{(n+1)\times(n+1)}$ is a positive definite and symmetric matrix; a common choice is $Q = I$, where I is the identity matrix of the appropriate size. Furthermore, a unique solution for Eq. (8), $P = P^T > 0$, always exists, as it has been assumed that A in Eq. (5) is Hurwitz.

Next, by performing calculations, we find that the time derivative of the Lyapunov function in Eq. (7) meets the following condition:

$$\dot{V}(\tilde{e}) \le -\left[\lambda_{min}(Q) - 2\lambda_{max}(P)\gamma\right]\left\|\tilde{e}\right\|_2^2, \tag{9}$$

where $\lambda_{min}(\bullet)$ and $\lambda_{max}(\bullet)$ represent the minimum and maximum eigenvalues, respectively.

If A is assumed to be Hurwitz, a sufficient condition for the local stability of the system in Eq. (5) is that the bound γ on the perturbation term in Eq. (6) is sufficiently small to meet the following requirement:

$$\gamma < \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)}. \tag{10}$$

As a result, the time derivative of the Lyapunov function is nonpositive, indicating that the error dynamics are asymptotically stable and that the master and slave systems achieve synchronization.

Although the aforementioned scheme can synchronize most pairs of systems, it is not universally applicable. For instance, the synchronization approach with a dynamic controller does not succeed with the Lorenz and Lü chaotic systems. To address this issue, we propose an improved synchronization scheme that incorporates dynamic controllers.

3. Ameliorated Synchronization Scheme Using Dynamic Controllers

Consider the master–slave system described below.

$$\textbf{Master}:\begin{cases}\dot{x}_m = F(x_m)\\ y_m = x_m\end{cases} \tag{11}$$

$$\textbf{Slave}:\begin{cases}\dot{x}_s = S(x_s) - Bh\\ y_s = Cx_s\end{cases} \tag{12}$$

$$\textbf{Dynamic controllers}: \dot{h} = -\alpha h - kC\left[x_m - x_s\right] \tag{13}$$

In this context, $x_m \in \mathbb{R}^n$ represents the state vector of the master system, whereas $x_s \in \mathbb{R}^n$ denotes the state vector of the slave system. The outputs of the two systems are $y_m \in \mathbb{R}$ and $y_s \in \mathbb{R}$, respectively. It is assumed that the functions F and S are sufficiently smooth, B is a constant column vector in $\mathbb{R}^n$, C is a constant row vector in $\mathbb{R}^{1\times n}$, h is the dynamic control input in $\mathbb{R}$, k is the coupling strength in $\mathbb{R}_+$, and α is a design parameter in $\mathbb{R}_+$.

Suppose that the nonlinear functions F and S comprise both linear and nonlinear components:

$$F(x_i) = Ax_i + f(x_i), \tag{14}$$

$$S(x_s) = F(x_s) - M(x_s), \tag{15}$$

where A is a constant matrix in $\mathbb{R}^{n\times n}$ and M is also a matrix in $\mathbb{R}^{n\times n}$ that can be used to adjust or modify the dynamic behavior of the slave system.

Thus, the error dynamics for the systems described in Eqs. (11)–(13) can be expressed as

$$\begin{pmatrix}\dot{e}\\ \dot{h}\end{pmatrix} = \begin{pmatrix}A & B\\ -kC & -\alpha\end{pmatrix}\begin{pmatrix}e\\ h\end{pmatrix} + \begin{pmatrix}E(t,e)\\ 0\end{pmatrix}, \tag{16}$$

where $e = x_m - x_s$ and $E(t,e) = f(x_m) - f(x_s) - M(x_s)$ is the state vector.

As the trajectories of the master system are bounded, the term $g(t,e) = f(x_m) - f(x_s)$ can be considered a perturbation. Thus, the error dynamics for the systems can be expressed as

$$\begin{pmatrix}\dot{e}\\ \dot{h}\end{pmatrix} = \begin{pmatrix}A & B\\ -kC & -\alpha\end{pmatrix}\begin{pmatrix}e\\ h\end{pmatrix} + \begin{pmatrix}M(t,e)\\ 0\end{pmatrix} + \begin{pmatrix}g(t,e)\\ 0\end{pmatrix}, \tag{17}$$

$$\bar{A} = \begin{pmatrix}A & B\\ -kC & -\alpha\end{pmatrix}\begin{pmatrix}e\\ h\end{pmatrix},\ \bar{m}(t,\tilde{e}) = \begin{pmatrix}M(t,e)\\ 0\end{pmatrix},\ \bar{g}(t,\tilde{e}) = \begin{pmatrix}g(t,e)\\ 0\end{pmatrix}, \tag{18}$$

where $\tilde{e} \in \mathbb{R}^{n+1}$ is the state vector, the matrix $\overline{A} \in \mathbb{R}^{(n+1)\times(n+1)}$ is assumed to be Hurwitz,[16] and $\overline{m}(t,\tilde{e})$ and $\overline{g}(t,\tilde{e})$ are the error and perturbation terms, respectively.

Finally, the error dynamics of the system can be expressed as

$$\begin{pmatrix} \dot{e} \\ \dot{h} \end{pmatrix} = \overline{A} \begin{pmatrix} e \\ h \end{pmatrix} + \overline{m}(t,\tilde{e}) + \overline{g}(t,\tilde{e}). \tag{19}$$

4. Synchronized Lorenz and Lü Chaotic Systems

The Lorenz chaotic system[17] is a famous example of chaotic behavior in dynamical systems. It is described by the following equations:

$$\begin{cases} \dfrac{dx}{dt} = \sigma(y-x), \\ \dfrac{dy}{dt} = x(\rho - z) - y, \\ \dfrac{dz}{dt} = xy - \beta z, \end{cases} \tag{20}$$

where $\sigma = 10$, $\beta = 8/3$, and $\rho = 28$.

The Lü chaotic system[18] is also an important model of 3D chaotic systems. It can be described as

$$\begin{cases} \dfrac{dx}{dt} = a(y-x), \\ \dfrac{dy}{dt} = -xz + cy, \\ \dfrac{dz}{dt} = xy - bz, \end{cases} \tag{21}$$

where $a = 36$, $b = 3$, and $c = 20$.

The Lorenz and Lü chaotic systems are both classic chaotic systems with high sensitivity and complex dynamic behavior. Now, we regard these two systems as master–slave systems, described as follows.

$$\textbf{Master}: \begin{cases} \dfrac{dx_m}{dt} = \sigma(y_m - x_m) \\ \dfrac{dy_m}{dt} = x_m(\rho - z_m) - y_m \\ \dfrac{dz_m}{dt} = x_m y_m - \beta z_m \end{cases} \tag{22}$$

$$\textbf{Slave}:\begin{cases}\dfrac{dx_s}{dt}=a\left(y_s-x_s\right)-h_x\\ \dfrac{dy_s}{dt}=-x_sz_s+cy_s-h_y\\ \dfrac{dz_s}{dt}=x_sy_s-bz_s-h_z\end{cases} \tag{23}$$

$$\textbf{Dynamic controllers}:\begin{cases}\dot{h}_x=-\alpha h_x-k\left[\left(x_m-x_s\right)\right]\\ \dot{h}_y=-\alpha h_y-k\left[\left(y_m-y_s\right)\right]\\ \dot{h}_z=-\alpha h_z-k\left[\left(z_m-z_s\right)\right]\end{cases} \tag{24}$$

When the error functions are defined as $e_1=x_m-x_s$, $e_2=y_m-y_s$, and $e_3=z_m-z_s$, the dynamics can be expressed in the form of Eq. (17) with

$$\begin{pmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\\ \dot{h}_x\\ \dot{h}_y\\ \dot{h}_z\end{pmatrix}=\begin{pmatrix}-\sigma & \sigma & 0 & 1 & 0 & 0\\ \rho & -1 & 0 & 0 & 1 & 0\\ 0 & 0 & -\beta & 0 & 0 & 1\\ -k & 0 & 0 & -\alpha & 0 & 0\\ 0 & -k & 0 & 0 & -\alpha & 0\\ 0 & 0 & -k & 0 & 0 & -\alpha\end{pmatrix}\begin{pmatrix}e_1\\ e_2\\ e_3\\ h_x\\ h_y\\ h_z\end{pmatrix}+\begin{pmatrix}\left(a-\sigma\right)x_s+\left(\sigma-a\right)y_s\\ \rho x_s-\left(c+1\right)y_s\\ \left(b-\beta\right)z_s\\ 0\\ 0\\ 0\end{pmatrix}+\begin{pmatrix}0\\ -x_mz_m+x_sz_s\\ x_my_m-x_sy_s\\ 0\\ 0\\ 0\end{pmatrix}, \tag{25}$$

where $\sigma = 10$, $\beta = 8/3$, $\rho = 28$, $a = 36$, $b = 3$, and $c = 20$.

$$\begin{pmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\\ \dot{h}_x\\ \dot{h}_y\\ \dot{h}_z\end{pmatrix}=\begin{pmatrix}-10 & 10 & 0 & 1 & 0 & 0\\ 28 & -1 & 0 & 0 & 1 & 0\\ 0 & 0 & -\frac{8}{3} & 0 & 0 & 1\\ -k & 0 & 0 & -\alpha & 0 & 0\\ 0 & -k & 0 & 0 & -\alpha & 0\\ 0 & 0 & -k & 0 & 0 & -\alpha\end{pmatrix}\begin{pmatrix}e_1\\ e_2\\ e_3\\ h_x\\ h_y\\ h_z\end{pmatrix}+\begin{pmatrix}26x_s-26y_s\\ 28x_s-21y_s\\ \left(\frac{1}{3}\right)z_s\\ 0\\ 0\\ 0\end{pmatrix}+\begin{pmatrix}0\\ -x_mz_m+x_sz_s\\ x_my_m-x_sy_s\\ 0\\ 0\\ 0\end{pmatrix} \tag{26}$$

According to the Rouih–Hurwitz stability criterion,[14] the characteristic polynomial of matrix $\overline{A}$ will have negative roots if and only if the following condition is satisfied:

$$k = 1600 \text{ and } 102.82772345116345 \le a < 135.27539822911677.$$

We selected and designed the parameters α and k in the control term to ensure that the chaotic system, with the control term applied, complies with the support of Lyapunov stability, thereby achieving chaotic synchronization. We can certainly find appropriate values for α and k to achieve chaotic synchronization.

As a result, the error dynamics of the systems in Eqs. (22)–(24) are globally asymptotically stable, meaning that the master and slave chaotic systems will synchronize asymptotically.

Figure 1 shows the changes in the 3D error equations over time. We start to add dynamic control from 1000 ms. The change in the curve in the figure shows the effect of dynamic control. After 1000 ms, the curve gradually becomes stable and close to zero, which shows that the two systems are synchronized.

5. Discussion

The synchronization method using a dynamic controller can successfully achieve synchronization in some cases where the traditional master–slave approach with a static controller fails. However, it is not universally applicable to all systems. To address this limitation, we propose an improved synchronization method that incorporates dynamic controllers. The strength of our proposed approach lies in enhancing the coupling between systems by increasing the dimensionality of the coupling. In certain systems, simply increasing the coupling strength within the same dimension may not be sufficient for synchronization. In such cases, the Rouche–Hurwitz stability criterion suggests that it is not possible to find appropriate values for the design parameter α and the coupling strength k that would make the error dynamics of the systems asymptotically stable. Our research includes an in-depth comparison of the number of dimensions in the control term. It focuses on improving the dynamic control synchronization method by transforming a system that originally cannot achieve synchronization into one that can, rather than comparing its performance with other different control methods.

To overcome this challenge and improve the overall coupling effect, our enhanced scheme boosts the coupling strength across multiple dimensions, effectively introducing coupling in two distinct dimensions simultaneously. This strategy results in the desired synchronization outcome.

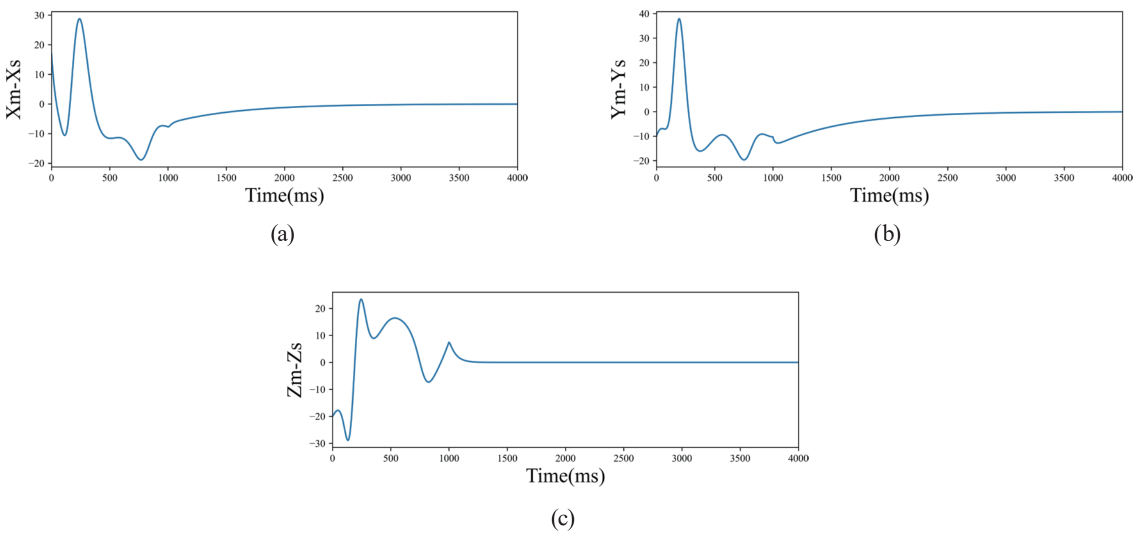

Fig. 1. (Color online) Time series of (a) $x_m - x_s$, (b) $y_m - y_s$, and (c) $z_m - z_s$.

6. Conclusions

In the pursuit of a more comprehensive and effective approach to synchronize all systems, we proposed an enhanced synchronization scheme that employs dynamic controllers. This novel scheme is specifically designed to address the synchronization challenges of chaotic systems, effectively synchronizing both the Lorenz and Lü chaotic systems, which were initially unsynchronized. The key innovation in our approach lies in the enhancement of the coupling mechanism, particularly by expanding the dimensionality of the coupling terms. Through this adjustment, the synchronization is achieved more efficiently and robustly. Our result validates the feasibility and effectiveness of this proposed synchronization method.

In this research, we primarily focused on addressing the challenges encountered in dynamic control synchronization—specifically, the inability to achieve synchronization between two different chaotic systems. We proposed a novel improvement to achieve chaotic synchronization and conducted a theoretical investigation. First, we employed Lyapunov stability to support the validity of our approach and used numerical simulation programs to verify its feasibility. The design of circuit experiments is the next phase of our research. We have already transformed the general equations of the chaotic system into circuit equations and are currently selecting appropriate circuit components to realize the physical implementation. Our goal is for the proposed improvement in dynamic control synchronization to gain theoretical recognition, providing a foundation for further implementation of actual circuit synchronization.

Moreover, the versatility of this approach suggests its potential application in developing advanced chaos-synchronization sensors, which can be instrumental for monitoring and controlling complex systems in various fields of science and engineering.

In our future research, we will conduct a detailed evaluation of computational complexity, including the computation time and resource consumption of the controller under different system dimensions. Additionally, we will explore ways to further reduce computational costs while maintaining fast synchronization.

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