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Parametric Order Reduction of Proportionally Damped Second-Order Systems

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In this paper, the structure-preserving order reduction of proportionally damped and undamped second-order systems is presented. The discussion is based on the second -order Krylov subspace method, and it is shown that for systems with proportional damping, the damping matrix does not contribute to the projection matrices, and that the reduction can be carried out using the classical Krylov subspaces. As a result of direct projection, the reduced-order model is parameterized in terms of the damping coefficients.

1. Introduction

Model order reduction based on Krylov subspaces(1,2) was originally developed for the reduction of first-order systems (state-space). However, in engineering, it is often necessary to deal with second-order systems as they are results of common modern modeling techniques.(3–5) The application of Krylov subspace methods to a second-order system requires a transformation to a first-order one (linearization), which is undesirable since the structure of the original system is destroyed during model reduction.

In refs. 6 and 7, Krylov subspaces were used to reduce second-order systems while preserving their structure. This approach has been revisited by different authors in recent years proposing alternatives and some improvements.(8,9) Also, special model-reduction methods based on second-order Krylov subspaces have been developed to treat second-order systems directly.(10–13)

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In the present paper, by considering a special class of second-order systems, namely proportionally damped systems, a simplified alternative to the structure-preserving order reduction of second-order systems is suggested.

The large-scale models considered here are assumed to be given in the form,

$$\begin{cases} \mathbf{M}\ddot{\mathbf{z}}(t) + \mathbf{D}\dot{\mathbf{z}}(t) + \mathbf{K}\mathbf{z}(t) = \mathbf{G}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{L}\mathbf{z}(t), \end{cases} \tag{1}$$

where $\mathbf{M} \in \mathbb{R}^{n\times n}$, $\mathbf{D} \in \mathbb{R}^{n\times n}$, $\mathbf{K} \in \mathbb{R}^{n\times n}$, $\mathbf{G} \in \mathbb{R}^{m\times n}$, and $\mathbf{L} \in \mathbb{R}^{p\times n}$ are given constant matrices, and $\mathbf{u} \in \mathbb{R}^{m}$, $\mathbf{y} \in \mathbb{R}^{p}$, and $\mathbf{z} \in \mathbb{R}^{n}$ are the inputs, outputs, and states of the system, respectively. For mechanical systems, **M**, **D**, and **K** represent, respectively, the mass, damping, and stiffness matrices, $\mathbf{u}(t)$ corresponds to the vector of external forces, **G** is the input matrix, **y** is the output measurement vector, **L** is the output matrix, and **z** is the vector of internal generalized coordinates. For SISO systems, $p = m = 1$, and the matrices **G** and **L** change to the vectors **g** and $\mathbf{l}^T$.

In addition, in this paper, it is assumed that the damping is proportional, i.e., $\mathbf{D} = \alpha\mathbf{M} + \beta\mathbf{K}$, which is widely assumed in engineering.(14) In practice, the coefficients α and β are chosen on the basis of experimental results and previous experience and can vary for the same structural model depending on the external conditions. These facts pose an additional requirement for simulation, that is, the free variation of α and β without having to repeat the reduction procedure.

One of the reasons for its widespread usage is that proportional damping does not change the eigenspace of the original undamped problem.(14) For instance, when one uses the mode superposition method for reduction of a model with proportional damping,(15) then:

1. modal analysis is performed for the original undamped system,
2. a few most important modes are selected,
3. α and β are used as parameters for the reduced system.

On the basis of this idea, in refs. 16 and 17, a model-reduction method that preserves α and β as parameters has been suggested. It consists of first performing a Krylov-based model reduction for the original undamped model, and then applying the generated projection matrices to the proportionally damped system to obtain the final reduced model.

The validity of the approach has been empirically shown in refs. 16 and 17, however, without any mathematical proof of moment matching. One of the goals of this paper is to bridge this gap by providing this missing proof: It is shown that the projection matrix used for the reduction of a proportionally damped system by moment matching is independent of the damping matrix, and thus, the parameters α and β.

The paper is organized as follows. In the following section, the second-order Krylov subspace method is reviewed, and in §4, the order reduction of proportionally damped systems is discussed. The reduction of undamped systems is the focus of §5, and in §6.2, the proposed approach is applied to reduce a technical system.

2. Order Reduction Using Krylov Subspaces

In this section, model-order reduction using Krylov subspaces is introduced, together with the definitions of moments, Markov parameters, and Krylov subspace.

2.1 *Moments and Markov parameters*

We consider the dynamical MIMO system of the form:

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \end{cases} \tag{2}$$

where $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{N\times N}$, $\mathbf{B} \in \mathbb{R}^{N\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times N}$ are constant given matrices, $\mathbf{u}(t) \in \mathbb{R}^{m}$, $\mathbf{y}(t) \in \mathbb{R}^{p}$, $x(t) \in \mathbb{R}^{N}$ are respectively the input, output and states vectors of the system. For SISO systems, $p = m = 1$, the matrices **B** and **C** become the vectors **b** and $\mathbf{c}^T$, and the vectors **u** and **y** become the scalars u and y.

The transfer function of the system (eq. (2)) in the Laplace domain is:

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \tag{3}$$

For any MIMO system in the form (eq. (2)), the moments, which are in the form of $p\times m$ matrices and defined as the negative coefficients of the Taylor series expansion about zero of the system's transfer function, are calculated as follows:[(2)]

$$\mathbf{m}_i = \mathbf{C}(\mathbf{A}^{-1}\mathbf{E})^{-i}\mathbf{A}^{-1}\mathbf{B} \qquad i = 0,1,\cdots. \tag{4}$$

These moments, which reflects the system behavior at low frequencies, become the scalars m_i for the case of SISO system.

In addition, moments can be defined about points $s_0 \neq 0$, reflecting the behavior of the system at different higher frequencies, by simply rewriting the transfer function of (eq. (3)) as:

$$\mathbf{H}(s) = \mathbf{C}[(s - s_0)\mathbf{E} - (\mathbf{A} - s_0\mathbf{E})]^{-1}\mathbf{B} \tag{5}$$

By comparing (eq. (3)) and (eq. (5)), it can be easily seen that the moments about s_0 can be computed by replacing **A** by $(\mathbf{A} - s_0\mathbf{E})$ in (eq. (4)), assuming that $(\mathbf{A} - s_0\mathbf{E})$ is nonsingular.

Special parameters known as Markov parameters that reflect the behavior at very high frequencies are defined when s_0 in (eq. (5)) tends to infinity ($s_0 \to \infty$):[(2)]

$$\mathbf{M}_i = \mathbf{C}(\mathbf{E}^{-1}\mathbf{A})^{-i}\mathbf{E})^{-1}\mathbf{B} \qquad i = 0,1,\cdots. \tag{6}$$

Note that the *i*th Markov parameter constitutes the *i*th derivative of the impulse response of the system (eq. (2)) at time zero,(2) and consequently the first Markov parameter $\mathbf{M}_0$ is the impulse response at $t = 0$.

The moments (around $s_0 = 0$, and $s_0 \neq 0$) and the Markov parameters will be used to describe the similarity between the original and reduced order models in the model-order reduction approach using Krylov subspaces based on the following facts:

1. With $s_0 = 0$, the reduced and original model have the same DC gain and steady state accuracy is achieved.
2. Small values of s_0 find a reduced model with good approximation of the slow dynamics.
3. Large values of s_0 and/or matching the Markov parameters find a reduced model with good approximation at high frequencies.
4. It is possible to simultaneously match the moments about different frequency points s_0 (and Markov parameters) to achieve better approximation on a wider frequency band or on a specific frequency band of interest. For complex mechanical systems with mixed mode vibration, this would result in a reduced-order model approximating simultaneously the dominant low- and high-frequency modes.

2.2 *Krylov subspace*

The Krylov subspace is defined as,

$$\mathcal{K}_Q(\mathbf{A}_1, \mathbf{B}_1) = span\left\{\mathbf{B}_1, \mathbf{A}_1\mathbf{B}_1, \ldots, \mathbf{A}_1^{Q-1}\mathbf{B}_1\right\}$$

where $\mathbf{A}_1 \in \mathbb{R}^{N\times N}$, and the columns of the matrix $\mathbf{B}_1 \in \mathbb{R}^{N\times m}$ are called starting vectors. For order reduction purposes and considering the state-space system (eq. (2)), two Krylov subspaces $\mathcal{K}_{Q_1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{B})$ and $\mathcal{K}_{Q_2}(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{A}^{-T}\mathbf{C}^T)$ are defined, called input and output Krylov subspaces, respectively.(2,13)

Now, if the projection

$$\mathbf{x} = \mathbf{V}\mathbf{x}_r,\ \mathbf{V} \in \mathbb{R}^{N\times Q},\ \ \mathbf{x} \in \mathbb{R}^N,\ \mathbf{x}_r \in \mathbb{R}^Q, \tag{7}$$

with $Q < N$, is applied to the system (eq. (2)) and after multiplying the state equation by the transpose of a matrix $\mathbf{W} \in \mathbb{R}^{N\times Q}$, the following reduced model is obtained:

$$\begin{cases} \overbrace{\mathbf{W}^T\mathbf{E}\mathbf{V}}^{\mathbf{E}_r}\,\dot{\mathbf{x}}_r(t) = \overbrace{\mathbf{W}^T\mathbf{A}\mathbf{V}}^{\mathbf{A}_r}\,\mathbf{x}_r(t) + \overbrace{\mathbf{W}^T\mathbf{B}}^{\mathbf{B}_r}\,u(t), \\ \mathbf{y}(t) = \underbrace{\mathbf{C}\mathbf{V}}_{\mathbf{C}_r}\,\mathbf{x}_r(t), \end{cases} \tag{8}$$

It can be proved that, if the columns of $\mathbf{V}$ form a basis for the input or output Krylov subspace, then the first $\frac{Q}{m}$ or $\frac{Q}{p}$ moments of the original and reduced models match and the method is called a one-sided Krylov method. When the columns of both $\mathbf{V}$ and $\mathbf{W}$

form bases for the input and output Krylov subspaces respectively, then the number of matched moments is increased to $\frac{Q}{m} + \frac{Q}{p}$; this method is called a two-sided Krylov method.(1,2,13) A typical choice that has some advantages in preserving stability(2,13) in the one-sided Krylov method is **W** = **V**.

For the computation of the matrices **V** and **W** the Lanczos, Arnoldi and two-sided Arnoldi algorithms and their numerous improvements and modified versions are used. For more details see refs. 2 and 13 and the references therein.

For mechanical engineers, mode superposition (or modal approximation) is a basic model reduction tool, that is based on the assumption that only a few modes are necessary to describe the body dynamics. Mathematically, by decomposing the transfer function in terms of partial fractions, or by using the physical insight to the system, this approach obtains the reduced system by preserving some of the dominant poles, i.e., the eigenvalues of **A** close to the imaginary axis). Even though this method works well for many applications, it can be shown that it may fail when the eigenvalue decomposition contains Jordan blocks. On the other hand, model reduction using Krylov subspaces implicitly preserves dominant poles. One can even say that the reduced-order model, obtained by this approach, automatically inherit the dominant poles from the original system.

3. Order Reduction of Second-Order Systems

The reduction approach considered in this paper is also based on matching the moments and/or the Markov parameters, however for second-order systems of the form (eq. (1)). In this section, this order reduction approach that employs second-order Krylov subspaces will be reviewed, together with the required mathematical background.

The second-order model (eq. (1)) can be rewritten in state-space as:

$$\begin{cases} \underbrace{\begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}}_{\mathbf{E}} \underbrace{\begin{bmatrix} \dot{\mathbf{z}}(t) \\ \ddot{\mathbf{z}}(t) \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{F} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{G} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}(t), \\ \mathbf{y}(t) = \underbrace{\begin{bmatrix} \mathbf{L} & \mathbf{0} \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix}, \end{cases} \tag{9}$$

with the matrices **E**, **A**, **B**, and **C**, and the vectors **x** and $\dot{\mathbf{x}}$ introduced, to show the similarity with the conventional state-space representation model of eq. (2). As the dimension of the state **z** of the original second-order system (eq. (1)) is assumed to be equal to n, the order of its corresponding linearized state-space model (eq. (9)) (also called McMillan degree of $\mathbf{H}(s)$) is equal to $N = 2n$, i.e., the equivalent model (eq. (1)) has N first order differential equations. The matrix $\mathbf{F} \in \mathbb{R}^{n \times n}$ is a nonsingular optional matrix. As the choice of **F** does not have any effect on the upcoming facts and results, and for simplicity, **F** can be chosen to be the identity matrix **I**.

Now, assuming that **K** is nonsingular, the moments (about zero) of the equivalent system (eq. (9)) are,

$$\mathbf{m}_i = \mathbf{C}(\mathbf{A}^{-1}\mathbf{E})^i\mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix}\mathbf{0} & -\mathbf{L}\mathbf{K}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{0} & -\mathbf{M}\mathbf{K}^{-1} \\ \mathbf{I} & -\mathbf{D}\mathbf{K}^{-1}\end{bmatrix}^i\begin{bmatrix}\mathbf{0} \\ \mathbf{G}\end{bmatrix}. \tag{10}$$

Similar to the state-space case of the previous section, with the aim of matching the moments defined in eq. (10), the original second-order system (eq. (1)) is reduced by applying a projection, resulting in the following reduced model,

$$\begin{cases}\mathbf{W}^T\mathbf{M}\mathbf{V}\ddot{\mathbf{z}}_r + \mathbf{W}^T\mathbf{D}\mathbf{V}\dot{\mathbf{z}}_r + \mathbf{W}^T\mathbf{K}\mathbf{V}\mathbf{z}_r = \mathbf{W}^T\mathbf{G}\mathbf{u}, \\ \mathbf{y} = \mathbf{L}\mathbf{V}\mathbf{z}_r.\end{cases} \tag{11}$$

To calculate the projection matrices $\mathbf{V} \in \mathbb{R}^{n\times q}$ and $\mathbf{W} \in \mathbb{R}^{n\times q}$, achieving moment matching, the second-order Krylov subspaces,(10) defined as,

$$\tilde{\mathcal{K}}_q(\mathbf{A}_1, \mathbf{A}_2, \mathbf{G}_1) = span\{\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_{q-1}\}, \tag{12}$$

$$\text{where} \begin{cases}\mathbf{P}_0 = \mathbf{G}_1, \mathbf{P}_1 = \mathbf{A}_1\mathbf{P}_0 \\ \mathbf{P}_i = \mathbf{A}_1\mathbf{P}_{i-1} + \mathbf{A}_2\mathbf{P}_{i-2}, i = 2,3, \ldots\end{cases} \tag{13}$$

where $\mathbf{A}_1$, $\mathbf{A}_2 \in \mathbb{R}^{n\times n}$, $\mathbf{G}_1 \in \mathbb{R}^{n\times n}$ are constant matrices, can be used. The columns of $\mathbf{G}_1$ are called the starting vectors and the matrices $\mathbf{P}_i$ are called the basic blocks. The subspaces $\tilde{\mathcal{K}}_{q1}(-\mathbf{K}^{-1}\mathbf{D}, -\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G})$ and $\tilde{\mathcal{K}}_{q2}(-\mathbf{K}^{-1}\mathbf{D}, -\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{L})$ are called the input and output second-order Krylov subspaces, for the system (eq. (1)), respectively.

Theorem 1. If the columns of the matrix $\mathbf{V}$ used in (eq. (11)) form a basis for the input or output second-order Krylov Subspace, and $\mathbf{W}$ is chosen such that $\mathbf{W}^T\mathbf{K}\mathbf{V}$ is nonsingular, then the first q_1 or q_2 moments of the original and reduced models match.

In one-sided methods, as mentioned in Theorem 1, $\mathbf{W} = \mathbf{V}$ is a typical choice that also has some advantages in preserving stability.(12,13) The number of matching moments can be increased to q_1+q_2 by using both, the input and output second-order Krylov subspaces, for the choices of $\mathbf{V}$ and $\mathbf{W}$.

In order to achieve a good approximation at higher frequencies, the moments about $s_0 \neq 0$ are to be matched. It can be shown that this is achieved by substituting the matrices $\mathbf{K}$ by $\mathbf{K}_{s_0} = \mathbf{K} + s_0\mathbf{D} + s_0^2\mathbf{M}$ and $\mathbf{D}$ by $\mathbf{D}_{s_0} = \mathbf{D} + 2s_0\mathbf{M}$, in the corresponding second-order Krylov subspace.(13) In this case, the condition of the nonsingularity of $\mathbf{K}$ is substituted by the nonsingularity of $\mathbf{K}_{s_0}$. In other words, s_0 should not be a quadratic eigenvalue of the triple $(\mathbf{M},\mathbf{D},\mathbf{K})$.

4. Proportionally Damped Systems

Let the second-order system (eq. (1)) be proportionally damped. First, it is shown how for this family of second-order systems, the second-order Krylov subspaces used for moment matching about zero can be reduced to the classical Krylov subspaces without affecting the moment-matching property.

Theorem 2. If $\mathbf{D} = \alpha\mathbf{M} + \beta\mathbf{K}$ with $\alpha \neq 0$ then,

$$\tilde{\mathcal{K}}_q(-\mathbf{K}^{-1}\mathbf{D}, -\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G}) = \mathcal{K}_q(-\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G}).$$

Proof: Let $\mathbf{P}_i$ and $\hat{\mathbf{P}}_i$ be the basic blocks of the second-order and standard Krylov subspaces, respectively. It is shown that the basic blocks of the two subspaces span the same space by proving that the *i*th basic block of one subspace can be written as a linear combination of the first *i* blocks of the other.

The starting vectors are clearly the same, $\mathbf{P}_0 = \hat{\mathbf{P}}_0$. For the next basic block, we have,

$$\begin{aligned}\mathbf{P}_1 &= \mathbf{K}^{-1}\mathbf{D}\mathbf{K}^{-1}\mathbf{G} = \mathbf{K}^{-1}(\alpha\mathbf{M} + \beta\mathbf{K})\,\mathbf{K}^{-1}\mathbf{G}\\ &= \alpha\mathbf{K}^{-1}\mathbf{M}\mathbf{K}^{-1}\mathbf{G} + \beta\mathbf{K}^{-1}\mathbf{G} = \alpha\hat{\mathbf{P}}_1 + \beta\hat{\mathbf{P}}_0\end{aligned}$$

Now consider that $\mathbf{P}_i = \sum_{j=0}^{i} c_j\hat{\mathbf{P}}_j$ for $i = 0, \cdots, k-1$. For $i = k$, we have,

$$\begin{aligned}\mathbf{P}_k &= -\mathbf{K}^{-1}\mathbf{D}\mathbf{P}_{k-1} - \mathbf{K}^{-1}\mathbf{M}\mathbf{P}_{k-2} = -\mathbf{K}^{-1}(\alpha\mathbf{M} + \beta\mathbf{K})\,\mathbf{P}_{k-1} - \mathbf{K}^{-1}\mathbf{M}\mathbf{P}_{k-2}\\ &= (-\alpha\mathbf{K}^{-1}\mathbf{M} - \beta)\sum_{j=0}^{k-2} c_j\hat{\mathbf{P}}_j - \mathbf{K}^{-1}\mathbf{M}\sum_{j=0}^{k-2} c_j\hat{\mathbf{P}}_j = \alpha\sum_{j=1}^{k} c_j\hat{\mathbf{P}}_j - \beta\sum_{j=0}^{k-1} c_j\hat{\mathbf{P}}_j + \sum_{j=1}^{k-1} c_j\hat{\mathbf{P}}_j.\end{aligned}$$

The proof is completed by induction.

Theorem 2 shows that the projection matrix $\mathbf{V}$ can be calculated using the conventional Krylov subspace and be applied directly to reduce the original second-order system. Here, it should be noted that during this work an alternative proof to Theorem 2, which does not use the second-order Krylov methods, has been published in ref. 18.

In the following, a special class of systems with damping proportional only to the stiffness is discussed.

Theorem 3. If $\mathbf{D} = \beta\mathbf{K}$ ($\alpha = 0$) then,

$$\tilde{\mathcal{K}}_q(-\mathbf{K}^{-1}\mathbf{D}, -\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G}) = \mathcal{K}_{\frac{q}{2}}(-\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G}).$$

Proof: Following the proof of Theorem 2, in this case $\mathbf{P}_i = -\beta\mathbf{P}_{i-1} - \mathbf{K}^{-1}\mathbf{M}\mathbf{P}_{i-2}$. The starting vectors are clearly the same. For the next blocks,

$$\begin{aligned}\mathbf{P}_1 &= -\beta\mathbf{P}_0 = -\beta\hat{\mathbf{P}}_0\\ \mathbf{P}_1 &= -\beta\mathbf{P}_1 - \mathbf{K}^{-1}\mathbf{M}\mathbf{P}_0 = \beta^2\hat{\mathbf{P}}_0 + \beta\hat{\mathbf{P}}_1.\end{aligned}$$

Now consider that $\mathbf{P}_{2i} = \sum_{j=0}^{i} c_j\hat{\mathbf{P}}_j$ and $\mathbf{P}_{2i+1} = \sum_{j=0}^{i} d_j\hat{\mathbf{P}}_j$ for $i = 0, \cdots, k-1$. For $i = k$, we have,

$$\begin{aligned}\mathbf{P}_{2k} &= -\beta\mathbf{P}_{2k-1} - \mathbf{K}^{-1}\mathbf{M}\mathbf{P}_{2k-2} = -\beta\sum_{j=0}^{k-1} d_j\hat{\mathbf{P}}_j - \mathbf{K}^{-1}\mathbf{M}\sum_{j=0}^{k-1} c_j\hat{\mathbf{P}}_j \\ &= -\beta\sum_{j=0}^{k-1} d_j\hat{\mathbf{P}}_j + \sum_{j=1}^{k} c_j\hat{\mathbf{P}}_j, \\ \mathbf{P}_{2k+1} &= -\beta\mathbf{P}_{2k} - \mathbf{K}^{-1}\mathbf{M}\mathbf{P}_{2k-1} = -\beta^2\sum_{j=0}^{k-1} d_j\hat{\mathbf{P}}_j - \beta\sum_{j=1}^{k} c_j\hat{\mathbf{P}}_j - \mathbf{K}^{-1}\mathbf{M}\sum_{j=0}^{k-1} d_j\hat{\mathbf{P}}_j \\ &= -\beta^2\sum_{j=0}^{k-1} d_j\hat{\mathbf{P}}_j - \beta\sum_{j=1}^{k} c_j\hat{\mathbf{P}}_j + \sum_{j=1}^{k} d_j\hat{\mathbf{P}}_j.\end{aligned}$$

The proof is completed by induction showing that,

$$\begin{aligned}&span\left\{\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_{2k+1}\right\} = \\ &span\left\{\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_{2k}\right\} \subset span\left\{\hat{\mathbf{P}}_0, \hat{\mathbf{P}}_1, \ldots, \hat{\mathbf{P}}_k\right\}.\end{aligned}$$

Theorem 3 shows also that if $\alpha = 0$, deleting all the odd-indexed basic blocks from the second-order Krylov subspace does not affect the subspace and reduces by half the dimension of the reduced system while still matching the same number of moments. From the preceding two theorems, it is also remarked that the projection matrices are independent of the damping matrix, and thus of α and β.

These results can be generalized for the case of matching the moments about $s_0 \neq 0$. The system matrices involved in the calculation of the second-order Krylov subspaces for this case are:

$$\mathbf{M}_{s_0} = \mathbf{M}, \tag{14}$$

$$\mathbf{D}_{s_0} = \beta\mathbf{K} + (2s_0 + \alpha)\mathbf{M}, \tag{15}$$

$$\mathbf{K}_{s_0} = (1 + s_0\beta)\mathbf{K} + (s_0^2 + s_0\alpha)\mathbf{M}. \tag{16}$$

By manipulation of eqs. (14)–(16), the matrix $\mathbf{D}_{s_0}$ can be rewritten as $\gamma\mathbf{K}_{s_0} + \lambda\mathbf{M}_{s_0}$ with

$$\gamma = \frac{\beta}{1 + s_0\beta}, \quad \lambda = 2s_0 + \alpha + \frac{\beta(s_0^2 + s_0\alpha)}{1 + s_0\beta}.$$

Thus, the modified system is still proportionally damped, and using Theorem 2,

$$\tilde{\kappa}_q(-\mathbf{K}_{s_0}^{-1}\mathbf{D}_{s_0}, -\mathbf{K}_{s_0}^{-1}\mathbf{M}_{s_0}, -\mathbf{K}_{s_0}^{-1}\mathbf{G}) = \kappa_q(-\mathbf{K}_{s_0}^{-1}\mathbf{M}_{s_0}, -\mathbf{K}_{s_0}^{-1}\mathbf{G}). \tag{17}$$

A convenient choice of the expansion point for this class of systems would be $s_0 = -\frac{\alpha}{2}$, because it results in damping proportional only to the stiffness and makes Theorem 3 applicable here, consequently reducing the dimension of the reduced model by half.

5. Undamped Systems

In this section, the second-order system (eq. (1)) is considered to be undamped and thus, $\mathbf{D}$ = 0. The second-order Krylov subspace for moment matching about zero for undamped systems is $\tilde{\mathcal{K}}_q(\mathbf{0}, -\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G})$, resulting in the following projection matrix,

$$\begin{aligned} span(\mathbf{V}) &= span\{ -\mathbf{K}^{-1}\mathbf{G}, \mathbf{0}, \mathbf{K}^{-1}\mathbf{M}\mathbf{K}^{-1}\mathbf{G}, \mathbf{0}, \cdots\} \\ &= \mathcal{K}_{\frac{q}{2}}(-\mathbf{K}^{-1}\mathbf{M}, -\mathbf{K}^{-1}\mathbf{G}). \end{aligned}$$

This fact is also clear from Theorem 3 by setting β = 0.

It is well known, on the basis of eqs. (10) and (12), that the moments can be expressed as a function of the basic blocks $\mathbf{P}_i$ of the second-order Krylov subspace. As a consequence, the odd-indexed moments are zero as their corresponding $\mathbf{P}_i$ are zero. This fact can also be observed by a direct calculation of the moments as follows:

$$\mathbf{m}_i = \begin{bmatrix} \mathbf{L} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & -\mathbf{K}^{-1}\mathbf{M} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}^i \begin{bmatrix} -\mathbf{K}^{-1}\mathbf{G} \\ \mathbf{0} \end{bmatrix}. \tag{18}$$

By simple matrix manipulation, it is clearly seen that for all odd i, the corresponding moment is zero.

An advantage of applying a direct projection for the reduction of second-order systems is that if the original model is undamped, the reduced system is undamped too. This is helpful to match the zero moments automatically, as stated by the following remark:

Remark 1. If an undamped second-order system is reduced to dimension q, by a one-sided method ($\mathbf{W} = \mathbf{V} \in \mathbb{R}^{n\times q}$), then $2q$ moments match, with the odd-indexed moments among them being equal to zero.

When matching the moments about $s_0 \neq 0$ for an undamped system, the involved matrices in the calculations of the second-order Krylov subspaces are the following:

$$\begin{aligned} &\mathbf{M}_{s_0} = \mathbf{M}, \quad \mathbf{D}_{s_0} = 2s_0\mathbf{M}, \\ &\mathbf{K}_{s_0} = \mathbf{K} + s_0^2\mathbf{M}. \end{aligned}$$

It can be clearly seen that this is simply a special case of proportional damping with the matrix $\mathbf{D}_{s_0}$ proportional only to the matrix $\mathbf{M}_{s_0}$. Hence, the Krylov subspace of eq. (17) should be used for the reduction procedure.

6. Experimental Results and Discussion

A clamped-beam model has been used to demonstrate the effectiveness of the suggested method. The advantages are that the model is already well documented in ref. 19 and is available in the computer readable format.* The mechanical model shown in Fig. 1 is a typical structure whose generic layout corresponds to, e.g., atomic-force-microscopy tips, and gas sensors, as well as radio-frequency (RF) switches and filters. The real-life system consists of a beam and a counter electrode placed below it. A voltage source generates a potential difference between the two electrodes, which creates an attraction force between them and therefore results in a deformation of the flexible beam. In the considered model, this attraction force, shown in Fig. 1 as −F(t), is applied directly to the right end of the beam. Note that with these assumptions, only the mechanical problem has been considered.

6.1 *Modeling of the elastic beam*

The model of the beam is extracted under some approximations such as numerical discretization, constraints on the degrees of freedom, and material properties.[19]

The infinite dimensional partial differential equation is then approximated by an ordinary differential equation through finite-element discretization by splitting the beam into finite length elements. Obviously, the bigger the number of those elements is, the higher the order of the corresponding model becomes. Since the aspect ratio of a beam (i.e., the ratio of the length to the transverse dimensions) is rather large, we can further approximate the three-dimensional (3D) body of the beam by a one-dimensional (1D) curve embedded in 3D. For symmetry reasons, the beam motion can be constrained to a plane, yielding a two-dimensional (2D) motion. In this case, three possible beam deflections can be observed:[20]

- Torsional displacements: A rotation about the longitudinal axis of the beam.
- Axial displacements: Compression or expansion of the beam along its longitudinal axis.
- Flexural displacements: Deflection of the beam out of its plane undeformed axis.

It is assumed that the beam deflection is small, so that geometric nonlinearities can be neglected. This allows us to impose another constraint on the beam motion: x and y deflection are decoupled. It is also assumed that the possible deflections are smaller than the distance between the beams, so that no contact occurs. The material used is assumed to be isotropic and ideally elastic with no plastic deformation or brittle fracture. As is common in micromechanics, gravity may be neglected. For the numerical values of the beam dimensions and material properties, please refer to Table 1.

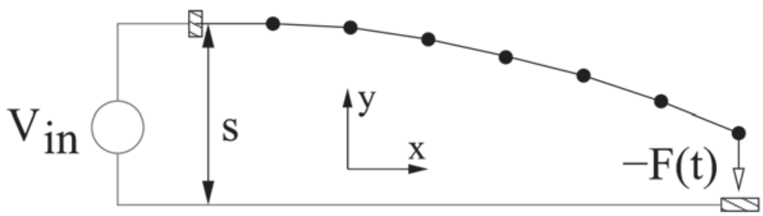

Fig. 1. Conducting clamped beam with counter electrode below right end.

*This model can be downloaded from the Oberwolfach Model Reduction Benchmark collection, at http://www.imtek.de/simulation/benchmark/index.php.

Table 1
Geometrical and material properties of the beam.

Parameter	Value
Beam Length (m)	0.1
Material density (rho)	8000
Cross-sectional area (A)	7.854×10^{-7}
Moment of inertia (I)	4.909×10^{-14}
Polar moment of inertia (J)	9.817×10^{-14}
Modulus of elasticity (E)	2 × 1011 Pa
Poisson ratio (nu)	0.29

A Lagrangian formulation is used to determine the equations of motion following the treatment in ref. 20 to calculate the energies. To represent the energy dissipation in the device, several damping effects should be considered. In fact, for this example, two major damping effects are to be included: the gas-damping that should include the gas dynamics, and the intrinsic damping that is due to the losses inside the material of the microstructure (internal friction and thermo-elastic damping). Unfortunately, a general and accurate description of damping is therefore very complex. In this paper, an assumption common to civil and mechanical engineers is adopted, that is, Rayleigh mode-preserving damping, i.e., the damping matrix **D** is calculated by a linear combination of the stiffness and mass matrices, resulting in $\mathbf{D} = \alpha\mathbf{M}+\beta\mathbf{K}$. Even though this assumption has often no physical background, it greatly simplifies the problem description and solution. The damping coefficients α and β are considered as effective values and are extracted based on the comparison between the experimental and the simulation dynamics.

6.2 *Reduced-order modeling of the beam*

We consider a model of order N = 15992 with n = 7996 second-order differential equations and proportional damping. The input to the system is the voltage between the electrodes, which is typically a step function or periodic on/off switching. The output of the system is the state number 5996, which is the displacement of a point on the last two-thirds of the beam. The reduction is carried out for two cases: the undamped case and a damped model with $\alpha = 100$, $\beta = 10^{-7}$.

The original model is reduced to different orders by the proposed approach of this paper and by applying the second-order Krylov method. In Table 2, the elapsed times of the two approaches are compared when reducing the model to order 3.

Since the proposed approach is independent of damping, less calculation is required in each step and the calculation is faster. Even if the damping is zero, to match the moments about zero, the second-order Krylov method requires double the number of iterations, since half of the columns of the projection matrix should be deleted. Furthermore, the new approach finds the projection matrix only once, whereas for the second-order Krylov method, it should be calculated separately for every choice of damping coefficients.

The step and frequency responses of the reduced systems for the undamped case are shown in Figs. 2 and 3, respectively.

The simulation results of the proportionally damped model can be seen in Figs. 4 and 5.

As the typical input to this system is a step response or a periodic on/off switching, the reduced model should thus both represent the step response as well as the possible influence of higher order harmonics. In Figs. 3 and 5, it can be seen that the step response is approximated very well by a reduced model of order $Q = 6$ ($q = 3$). In fact,

Table 2
Comparison between the reduced-order models.

Damping	Order $Q = 2q$	Method	Elapsed time
Undamped	6	Proposed approach	0.037 s
Undamped	6	Second order Krylov	0.26 s
Damped	6	Proposed approach	0.037 s
Damped	6	Second order Krylov	0.18 s

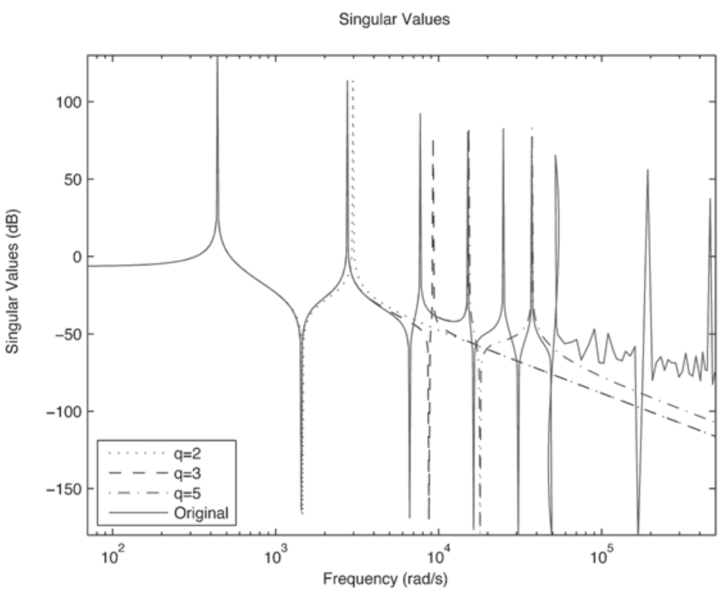

Fig. 2. Frequency response of the reduced systems of the undamped model.

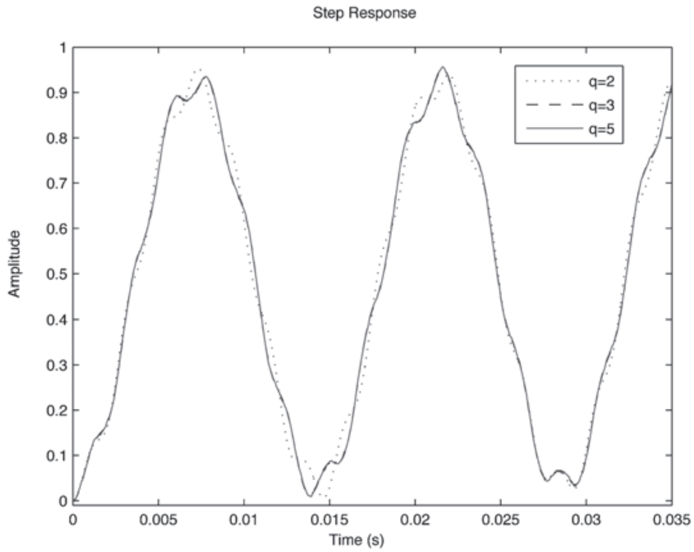

Fig. 3. Step response of the reduced systems of the undamped model.

by increasing the order, the step response changes very slightly. This is due to the fact that order reduction using Krylov subspaces automatically selects the best modes, in this case, only three are enough to approximate the original model. When compared to modal approximation, it can be remarked, that the latter method is more physical in finding the dominant modes, however the suggested method is more formal and thus automatic.

In the frequency response, note that by going to higher orders, better accuracy at higher frequencies is achieved. Because the second-order structure is preserved, the slope of the Bode plots at high frequencies is −40 dB/decade.

For this numerical example, the lowest eigenfrequency is dominant, however the method has been successfully applied to more complex mechanical models with mixed-mode vibration, by matching the moments and/or Markov parameters around different high and low frequency points. Application examples include among others, model reduction of a microaccelerometer,(16) a bonded wire,(17) and a Butterfly Gyroscope.(21)

7. Conclusions

In this paper, it was proved that if the damping of the original second-order system is proportional to the mass and stiffness matrices, the projection matrix can be calculated using the standard Krylov subspace.

The proposed approach not only preserves the second-order structure, but also carries out all calculations in the dimension of the second-order system without going to

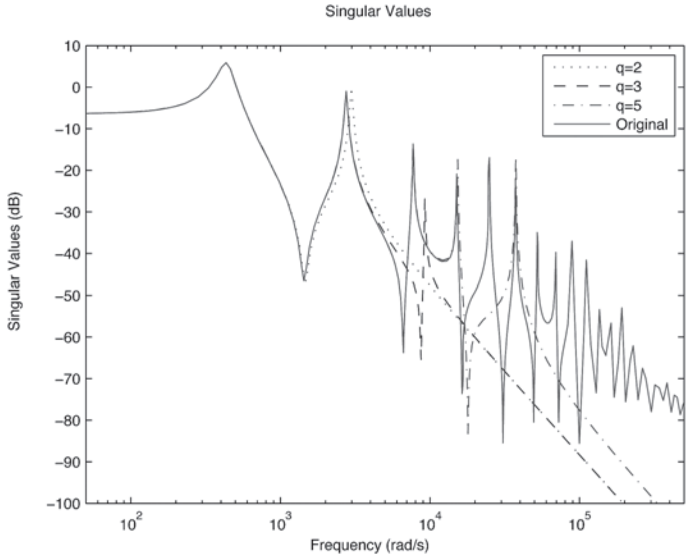

Fig. 4. Frequency response of the reduced systems of the damped model.

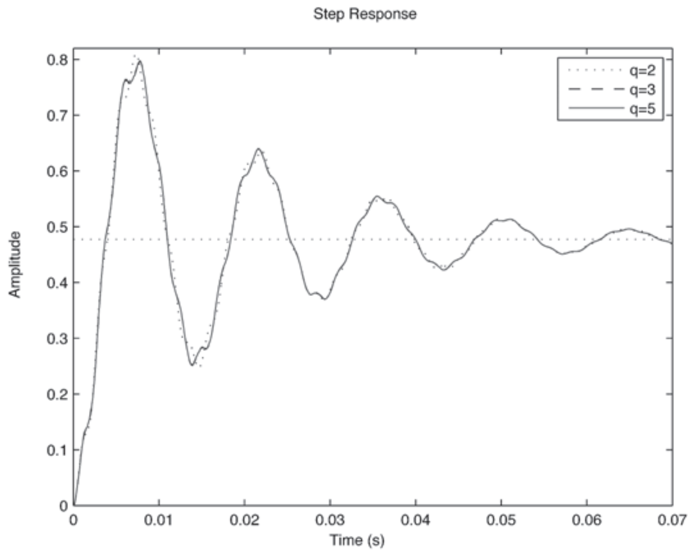

Fig. 5. Step response of the reduced systems of the damped model.

Table 3
Number of matched moments when reducing using a one-sided method (**W**= **V**) to order 2*q*.

System	α	s_0	Matched Moments
Undamped	–	0	2q
(**D** = 0)	–	≠0	q
Prop. Damped	0	0	2q
(**D** = α**M**+ β**K**)	≠0	0	q
	≠0	≠0	q

tatespace. It is stressed that in the case of proportional damping, the projection matrices, which guarantee moment matching, are calculated independently of α and β, unlike in the second-order Krylov methods, which require a new calculation of the projection matrices each time the damping coefficients are changed. In other words, despite the theoretical and computational simplicity of the new method compared with the methods of refs. 10–12, it still allows us to achieve exactly the same results and to preserve α and β as parameters in the symbolic form during the reduction procedure. Furthermore, because it is based on the classical Krylov subspaces, the proposed method takes advantage of the numerical reliable and well-known algorithms of standard Krylov subspace method such as Arnoldi and Lanczos to calculate the required projection matrices.

Finally, in Table 3, the number of matching parameters for the different cases discussed in this paper is summarized.

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